ON DONALDSON'S FLOW OF SURFACES IN A HYPERKÄHLER FOUR-MANIFOLD

Jian Song
Johns Hopkins University
Department of Mathematics
Baltimore, MD 21218

Ben Weinkove[1]
Imperial College
Department of Mathematics
London SW7 2AZ, U.K.

Abstract. We prove some basic properties of Donaldson's flow of surfaces in a hyperkähler 4-manifold. When the initial submanifold is symplectic with respect to one Kähler form and Lagrangian with respect to another, we show that certain kinds of singularities cannot form, and we prove a convergence result under a condition related to one considered by M.-T. Wang for the mean curvature flow.

1. Introduction

In [Do], Donaldson introduced a number of geometric evolution equations arising from the framework of moment maps and diffeomorphisms. If (S, ρ) and (M, ω) are two symplectic manifolds then a moment map can be constructed for the action of the group of symplectomorphisms of (S, ρ) on the space of smooth maps $f : S \to M$. The gradient flow of the square of the norm of the moment map gives, in many cases, a parabolic flow of maps. For example, when the two manifolds are of the same dimension and Kähler this gives rise to a flow of Kähler potentials known as the J-flow (see [Ch], [We], [SoWe]).

In this paper, we consider the case when S is a Riemann surface with volume form ρ and $(M, \overline{g}, I, J, K)$ a hyperkähler 4-manifold. M has three Kähler forms $\overline{\omega}_1$, $\overline{\omega}_2$ and $\overline{\omega}_3$, given by

$$\overline{\omega}_1(X,Y) = \overline{g}(IX,Y),\ \overline{\omega}_2(X,Y) = \overline{g}(JX,Y),\ \overline{\omega}_3(X,Y) = \overline{g}(KX,Y).$$

Given an immersion $f : S \to M$, define three functions on S by

$$N_a = \frac{f^*(\overline{\omega}_a)}{\rho}, \quad \text{for } a = 1, 2, 3. \tag{1.1}$$

Let ξ_a, for $a = 1, 2, 3$, be the Hamiltonian vector fields on S associated to N_a. Then Donaldson's flow is given by

$$\frac{\partial f}{\partial t} = If_*(\xi_1) + Jf_*(\xi_2) + Kf_*(\xi_3). \tag{1.2}$$

We will call this the *Hyperkähler mean curvature flow of Donaldson* or, for short, the H-flow. It is the gradient flow of the functional

$$E(f) = \sum_{a=1}^{3} \|N_a\|^2_{L^2(S,\rho)}.$$

[1]The second author is supported in part by National Science Foundation grant DMS-05-04285, and is currently on leave from Harvard University, supported by a Royal Society Research Assistantship

Donaldson showed that f is a critical point of E if and only if its image is a minimal surface and the induced volume form $d\mu$ on S is a constant multiple of ρ. He also proved that any compact immersed minimal surface in a hyperkähler 4-manifold with normal Euler number $e \geq -2$ which is not a complex curve for any complex structure on M must be not strictly stable with respect to E (see also the results of [MiWo]).

Donaldson showed that the H-flow is related to the mean curvature flow of surfaces in a 4-manifold. Indeed, define a function λ on S by

$$\lambda = \frac{d\mu}{\rho}.$$

Then the H-flow can be written

$$\frac{\partial f}{\partial t} = \lambda \nabla \lambda + \lambda^2 H, \tag{1.3}$$

where H is the mean curvature vector of S in M. This is proved in [Do]. We give another, very explicit, proof of this (Proposition 2.1).

The mean curvature flow of surfaces in a 4-manifold has been studied intensely over the last several years. Motivated by a question of Yau on how to deform a symplectic submanifold into a holomorphic one, Wang [Wa1] showed that if the initial submanifold is symplectic, it remains so along the mean curvature flow and that Type I singularities do not form (see also [ChLi1]). Convergence results have been obtained in a number of special cases [Wa1, Wa2], [TsWa], [ChLiTi]. Also, Lagrangian submanifolds remain Lagrangian along the mean curvature flow [Sm1]. This case has been of great interest in light of the SYZ conjecture [StYaZa] in Mirror Symmetry [ThYa] and there are a number of promising results [Sm2, Sm3], [SmWa], [Wa3], [ChLi2]. However, the Lagrangian mean curvature flow can exhibit some very bad behavior in general (see e.g. [Ne]).

The problem of finding necessary and sufficient conditions for the convergence of the mean curvature flow, even when Lagrangian or symplectic, is still very wide open. We hope that, as well as being of intrinsic interest, understanding the behavior of the H-flow may help us to understand this related problem.

We are mainly interested in the special case of the H-flow when the initial data is Lagrangian with respect to one Kähler form and symplectic with respect to another. The differential form $\theta = \overline{\omega}_2 + i\overline{\omega}_3$ is an I-holomorphic (2,0)-form on M. We consider the space $\mathcal{N}$ given by

$$\mathcal{N} = \{ f : S \to M \mid f \text{ is an immersion and } f^*(\theta) = \rho \}.$$

Donaldson [Do] gives a geometric argument to prove the following:

Theorem 1 *Let f_t be a solution of the H-flow for $0 \leq t \leq T$. If $f_0 \in \mathcal{N}$ then $f_t \in \mathcal{N}$ for $0 \leq t \leq T$.*

In section 3 we present a proof of this using the maximum principle. If an H-flow f_t lies in $\mathcal{N}$ we will call it a *Special H-flow*. This is a flow of Lagrangian submanifolds in M. One might hope that, at least under certain conditions, the Special H-flow should converge to a parametrised Special Lagrangian submanifold. These are elements

f of $\mathcal{N}$ with $e^{i\alpha}f^*(\overline{\omega}_1 + i\overline{\omega}_2) = d\mu$ for some constant angle α, or equivalently, with $\lambda =$ constant. In the case of hyperkähler 4-manifolds, these Special Lagrangian submanifolds are complex curves with respect to one of the complex structures.

An interesting property of the Special H-flow is that the induced area form $d\mu$ is bounded from above and below. Moreover, we have the following:

Theorem 2 *Let f_t be a solution of the Special H-flow on a maximal time interval $[0, T)$, for $0 \leq T \leq \infty$. Then*

(i) For $t \in [0, T)$, the induced area form $d\mu = d\mu(t)$ on S satisfies

$$\left(\inf_S \lambda|_{t=0}\right)\rho \leq d\mu \leq \left(\sup_S \lambda|_{t=0}\right)\rho.$$

(ii) If there exists a constant C_0 such that the norm squared of the second fundamental form $|A|^2$ satisfies

$$\sup_S |A|^2 \leq C_0, \qquad \textit{for } t \in [0, T),$$

then there exist constants C_k for $k = 1, 2, \ldots$ such that

$$\sup_S |\nabla^k A|^2 \leq C_k, \qquad \textit{for } t \in [0, T).$$

(iii) If $T < \infty$ then

$$\lim_{t \to T} \sup_S |A|^2 = \infty.$$

By analogy with the mean curvature flow [Wa1] we might expect that along the Special H-flow, Type I singularities do not form. A Type I singularity for the H-flow at a time T is a singularity satisfying the estimate

$$\sup_S |A|^2 \leq \frac{C}{T-t}, \quad \text{for } t < T,$$

for some constant C. We show that if there is such a singularity then the function λ must exhibit a certain 'bad' behavior. Precisely, we define a *Type I* singularity* at time T to be a Type I singularity with the additional property that for each $y_0 \in M$ there exist positive constants λ_0, δ and C' such that

$$|\lambda(x) - \lambda_0| \leq C'(d(f(x), y_0)^{2\delta} + (T-t)^\delta) \tag{1.4}$$

for all $(x, t) \in S \times [0, T)$, where d is the distance function in (M, g). Then we prove the following result.

Theorem 3 *Along the Special H-flow, Type I* singularities do not form.*

We believe that it should be possible to remove the Hölder-type condition (1.4) and rule out all Type I singularities.

Finally we prove that in a certain situation, similar to one considered by Wang [Wa1] for the mean curvature flow, the Special H-flow can indeed converge to a complex curve.

Theorem 4 *Suppose that $\overline{\zeta}$ is an anti-self-dual parallel calibrating form on M. Then there exists $\epsilon > 0$ such that if $f_0 \in \mathcal{N}$ satisfies*

$$\left(\frac{1}{\lambda}\right)|_{t=0} > 1 - \epsilon \quad \textit{and} \quad \frac{f_0^*\overline{\zeta}}{d\mu} > 1 - \epsilon,$$

then the Special H-flow exists for all time and converges in C^∞ to a map $f_\infty : S \to M$ such that $f_\infty(S)$ is a totally geodesic complex curve in M.

It is not difficult to find simple examples where these initial conditions are satisfied. Indeed, let S be a torus $\mathbb{C}/\Lambda$, for some lattice Λ, and let Ω be the symplectic form on S induced from the standard one on $\mathbb{C}$. Let $M = S \times S$ with the standard metric $\overline{g}$, coordinates $(y^1, \cdots, y^4)$ from $\mathbb{R}^4$ and the hyperkähler structure defined by

$$\begin{aligned}
\overline{\omega}_1 &= dy^1 \wedge dy^2 + dy^3 \wedge dy^4 \\
\overline{\omega}_2 &= dy^1 \wedge dy^4 + dy^2 \wedge dy^3 \\
\overline{\omega}_3 &= dy^1 \wedge dy^3 - dy^2 \wedge dy^4.
\end{aligned}$$

Set $\overline{\zeta} = dy^1 \wedge dy^3 + dy^2 \wedge dy^4$. Consider a diffeomorphism $\phi : S \to S$ preserving Ω and define $f_0 : S \to M$ by $f_0(x^1, x^2) = (x^1, \phi(x^1), x^2, \phi(x^2))$. Let $\rho = f_0^*(\overline{\omega}_2)$. Then $f_0^*(\overline{\omega}_3) = 0$ and, if ϕ is sufficiently close to the identity, f_0 satisfies the hypotheses of Theorem 4.

The outline of the paper is as follows: in section 2, some basic properties of the flow will be presented, including a derivation of (1.3); Theorem 1 is proved in section 3; a proof of Theorem 2 and the evolution of the second fundmental form and mean curvature are given in section 4; section 5 contains a proof of Theorem 3 using a monotonicity formula and a blow-up argument; finally, Theorem 4 is proved in section 6.

2. Basic properties of the flow

Before we derive formulas for this flow, we will describe some notation. Fix a point $p \in S$. We will write g for the induced metric $f^*\overline{g}$ on S. Let $\{x^1, x^2\}$ be a normal coordinate system for S at p and let $\{y^1, \cdots, y^4\}$ be a normal coordinate system for M at $f(p)$. We assume that the coordinate system $\{\partial_1, \partial_2\}$ at p is oriented so that

$$\rho(\partial_1, \partial_2) > 0.$$

We will use lower case letters $i, j, \ldots$ to denote the indices 1 and 2, Greek letters $\alpha, \beta, \ldots$ for the indices 3 and 4, and upper case letters $A, B, \ldots$ for the full range of indices 1,2,3,4. We will denote $\frac{\partial}{\partial x^i}$ by ∂_i and frequently identify it with $f_*\partial_i$. We assume that the $\{y^A\}$ are chosen so that $\frac{\partial}{\partial y^i} = f_*\partial_i$ at p, and hence $\frac{\partial f^A}{\partial x^i} = \delta_i^A$ at p. We will also pick a local orthonormal frame $\{e_3, e_4\}$ for the normal bundle.

Let $\overline{\nabla}$ and ∇ denote the Levi-Civita connections for $\overline{g}$ on M and g on S respectively. We use the superscripts T and N to denote the orthogonal projection of a vector in TM to the tangent and normal bundles of S respectively. At p, $(\overline{\nabla}_{\partial_i}\partial_j)^T = \nabla_{\partial_i}\partial_j = 0$.

The second fundamental form $A : TS \times TS \to NS$ is defined by $A(X,Y) = (\overline{\nabla}_X Y)^N$. Write

$$A(\partial_i, \partial_j) = h_{\alpha ij} e_\alpha, \quad \text{for} \quad h_{\alpha ij} = \overline{g}(e_\alpha, \overline{\nabla}_{\partial_i} \partial_j).$$

The mean curvature vector is given at p by $H = H_\alpha e_\alpha$, where $H_\alpha = g^{ij} h_{\alpha ij}$.

Recall now that the Hamiltonian vector fields ξ_a, for $a = 1, 2, 3$, are defined by

$$\rho(\xi_a, \cdot) = dN_a, \tag{2.1}$$

with the N_a defined by (1.1). Note that, calculating at p,

$$N_1^2 + N_2^2 + N_3^2 = \lambda^2 \left(({I_1}^2)^2 + ({J_1}^2)^2 + ({K_1}^2)^2 \right) = \lambda^2.$$

Writing $\xi_a = \xi_a^i \partial_i$ we have from (2.1),

$$\rho_{12} \xi_a^1 = \partial_2 N_a, \quad \rho_{12} \xi_a^2 = -\partial_1 N_a,$$

and hence, at p,

$$\xi_a = \lambda(\partial_2 N_a)\partial_1 - \lambda(\partial_1 N_a)\partial_2, \quad \text{for } a = 1, 2, 3.$$

Proposition 2.1 *The H-flow (1.2) can be written*

$$\frac{\partial f}{\partial t} = \lambda \nabla \lambda + \lambda^2 H. \tag{2.2}$$

Proof Calculate at a point p,

$$\begin{aligned}
If_* \xi_1 &= \lambda(\partial_2 N_1) I(\partial_1) - \lambda(\partial_1 N_1) I(\partial_2) \\
&= \lambda \partial_2 (\lambda \overline{g}(I(\partial_1), \partial_2)) I(\partial_1) - \lambda \partial_1 (\lambda \overline{g}(I(\partial_1), \partial_2)) I(\partial_2) \\
&= \lambda(\partial_2 \lambda) \overline{g}(\partial_2, I(\partial_1)) I(\partial_1) + \lambda(\partial_1 \lambda) \overline{g}(\partial_1, I(\partial_2)) I(\partial_2) \\
&\quad - \lambda^2 \overline{g}(\overline{\nabla}_{\partial_2} \partial_1, I(\partial_2)) I(\partial_1) - \lambda^2 \overline{g}(\overline{\nabla}_{\partial_1} \partial_2, I(\partial_1)) I(\partial_2) \\
&\quad + \lambda^2 \overline{g}(\overline{\nabla}_{\partial_1} \partial_1, I(\partial_2)) I(\partial_2) + \lambda^2 \overline{g}(\overline{\nabla}_{\partial_2} \partial_2, I(\partial_1)) I(\partial_1).
\end{aligned}$$

The same formula holds for $Jf_* \xi_2$ and for $Kf_* \xi_3$, replacing I wherever it occurs with J and K respectively. Now observe that for $i = 1$ or 2, $\{\partial_i, I(\partial_i), J(\partial_i), K(\partial_i)\}$ is an orthonormal basis for $T_p M$. Let X be any vector in $N_p S$. A short calculation shows that if we set

$$\begin{aligned}
Y &= \overline{g}(X, I(\partial_2)) I(\partial_1) + \overline{g}(X, I(\partial_1)) I(\partial_2) \\
&\quad + \overline{g}(X, J(\partial_2)) J(\partial_1) + \overline{g}(X, J(\partial_1)) J(\partial_2) \\
&\quad + \overline{g}(X, K(\partial_2)) K(\partial_1) + \overline{g}(X, K(\partial_1)) K(\partial_2),
\end{aligned}$$

then Y is orthogonal to ∂_1, $I(\partial_1)$, $J(\partial_1)$ and $K(\partial_1)$ and hence is zero. Set $X = \overline{\nabla}_{\partial_1} \partial_2 = \overline{\nabla}_{\partial_2} \partial_1$, and we see that the terms in $\frac{\partial f}{\partial t}$ involving X vanish (notice that in our coordinates $\nabla_{\partial_i} \partial_j = 0$ and so X is indeed orthogonal to $T_p S$). The remaining terms give

$$\begin{aligned}
If_*(\xi_1) + Jf_*(\xi_2) + Kf_*(\xi_3) &= \lambda(\partial_2 \lambda)\partial_2 + \lambda(\partial_1 \lambda)\partial_1 + \lambda^2 \overline{\nabla}_{\partial_1} \partial_1 + \lambda^2 \overline{\nabla}_{\partial_2} \partial_2 \\
&= \lambda \nabla \lambda + \lambda^2 H,
\end{aligned}$$

where we are again making use of the fact that $\{\partial_i, I(\partial_i), J(\partial_i), K(\partial_i)\}$ is an orthornormal basis for $T_p M$. Q.E.D.

A solution to this parabolic flow always exists for a short time.

Proposition 2.2 *Given any smooth initial map $f_0 \in \mathcal{M}$, there exists $T > 0$ such that (2.2) admits a unique smooth solution $f_t \in \mathcal{M}$ for $t \in [0, T)$.*

Proof It is well known that the mean curvature flow $\frac{\partial f}{\partial t} = H$ admits a unique smooth solution for a short time. Indeed, although the mean curvature flow is not strictly parabolic, it becomes so after modifying by a family of diffeomorphisms of S (for a detailed proof, see e.g. [Zh]). By the same argument, since λ is only first order in f, one can see that

$$\frac{\partial f}{\partial t} = \lambda^2 H,$$

admits a unique smooth solution for a short time. Define another family of diffeomorphisms $\phi_t : S \to S$ by

$$\frac{\partial \phi_t^i}{\partial t} = \lambda \nabla^i \lambda,$$

with $\phi_0 = \text{id}$. Then $\tilde{f}(x,t) = f(\phi(x,t), t)$ solves (2.2). Q.E.D.

We will now describe the evolution of the metric $g_{ij} = \overline{g}(\partial_i, \partial_j)$, the volume form $d\mu$ and λ along the H-flow.

Proposition 2.3 *Along the H-flow,*

(i) $\displaystyle \frac{\partial}{\partial t} g_{kl} = \nabla_k \nabla_l (\lambda^2) - 2\lambda^2 H_\alpha h_{\alpha kl}$

(ii) $\displaystyle \frac{\partial}{\partial t} d\mu = \left(\triangle (\frac{\lambda^2}{2}) - \lambda^2 |H|^2 \right) d\mu$

(iii) $\displaystyle \frac{\partial \lambda^2}{\partial t} = \lambda^2 \triangle \lambda^2 - 2(\lambda^2)^2 |H|^2,$

where, in (i), the repeated index α indicates that we are summing over an orthonormal frame for the normal bundle.

Proof We will work in our usual coordinate system at a point p. Then

$$\frac{\partial}{\partial t} g(\partial_k, \partial_l) = \overline{g}(\overline{\nabla}_{\nabla(\frac{\lambda^2}{2}) + \lambda^2 H} \partial_k, \partial_l) + \overline{g}(\partial_k, \overline{\nabla}_{\nabla(\frac{\lambda^2}{2}) + \lambda^2 H} \partial_l). \tag{2.3}$$

Note that by the torsion-free condition for $\overline{\nabla}$ and the equality

$$[\nabla(\frac{\lambda^2}{2}) + \lambda^2 H, f_* \partial_j] = f_* [\frac{\partial}{\partial t}, \partial_j] = 0,$$

we have

$$\begin{aligned}
\overline{g}(\overline{\nabla}_{\nabla(\frac{\lambda^2}{2}) + \lambda^2 H} \partial_k, \partial_l) &= \overline{g}(\overline{\nabla}_{\partial_k} (\nabla(\frac{\lambda^2}{2}) + \lambda^2 H), \partial_l) \\
&= \overline{g}(\frac{1}{2} \overline{\nabla}_{\partial_k} (\partial_i(\lambda^2) \partial_i) + \partial_k(\lambda^2) H + \lambda^2 \overline{\nabla}_{\partial_k} H, \partial_l) \\
&= \frac{1}{2} \partial_k \partial_l (\lambda^2) - \lambda^2 \overline{g}(H, \overline{\nabla}_{\partial_k} \partial_l) \\
&= \frac{1}{2} \partial_k \partial_l (\lambda^2) - \lambda^2 H_\alpha h_{\alpha kl}.
\end{aligned}$$

Substituting into (2.3) gives (i). (ii) and (iii) follow easily from (i). Q.E.D.

Notice that by the maximum principle, we immediately have:

Corollary 2.1 *Along the H-flow,*

$$\lambda(t) \leq \sup_S \lambda|_{t=0}.$$

3. Preservation of the condition $f^*(\theta) = \rho$ along the H-flow

In this section we will prove Theorem 1. Suppose that we have a smooth solution f_t of the H-flow for $t \in [0, T]$ for some $T > 0$ and suppose that the initial data is in $\mathcal{N}$. For $i = 1, 2, 3$, define functions η_i on S by

$$\eta_i = \frac{f^*\overline{\omega}_i}{d\mu}.$$

Then the condition $f^*(\theta) = \rho$ is equivalent to the equations

$$\eta_2 = 1/\lambda \quad \text{and} \quad \eta_3 = 0.$$

Set

$$Q = (\eta_2 - 1/\lambda)^2 + \eta_3^2.$$

We will show that there exists a constant C depending on f_t for $t \in [0, T]$ such that

$$\frac{d}{dt}Q \leq \lambda^2 \triangle Q + CQ. \tag{3.1}$$

The theorem will then follow, since by the maximum principle,

$$Q(t)e^{-Ct} \leq Q(0) = 0.$$

We will now prove (3.1). Let $\{x^1, x^2\}$ be a normal coordinate system for S centered at a point p as before and set

$$\nu_1 = \frac{K\partial_1 - \eta_3\partial_2}{\sqrt{1-\eta_3^2}}, \qquad \nu_2 = \frac{K\partial_2 + \eta_3\partial_1}{\sqrt{1-\eta_3^2}}.$$

Then $\{\partial_1, \partial_2, \nu_1, \nu_2\}$ gives an orthonormal basis for T_pM. Let $\xi_3 = \sqrt{1-\eta_3^2(p)}$. Then at the point p, in this basis, K is given by

$$K = \begin{pmatrix} 0 & \eta_3 & \xi_3 & 0 \\ -\eta_3 & 0 & 0 & \xi_3 \\ -\xi_3 & 0 & 0 & -\eta_3 \\ 0 & -\xi_3 & \eta_3 & 0 \end{pmatrix}.$$

Now set $a = \overline{\omega}_2(\partial_1, \nu_1)(p)$ and $\xi_2 = \overline{\omega}_2(\partial_1, \nu_2)(p)$ so that

$$\xi_2^2 = 1 - \eta_2^2 - a^2.$$

Then at the point p, J is given by

$$J = \begin{pmatrix} 0 & \eta_2 & a & \xi_2 \\ -\eta_2 & 0 & -\xi_2 & a \\ -a & \xi_2 & 0 & -\eta_2 \\ -\xi_2 & -a & \eta_2 & 0 \end{pmatrix}.$$

For $i, j = 1, 2$, set $h_{3ij} = \overline{g}(\nu_1, \overline{\nabla}_{\partial_i}\partial_j)$ and $h_{4ij} = \overline{g}(\nu_2, \overline{\nabla}_{\partial_i}\partial_j)$. Let $\overline{R}$ be the curvature of $\overline{g}$, given by

$$\overline{R}(X, Y)Z = \overline{\nabla}_X\overline{\nabla}_Y Z - \overline{\nabla}_Y\overline{\nabla}_X Z - \overline{\nabla}_{[X,Y]}Z.$$

We have the following lemma.

Lemma 3.1 *Let $\overline{\omega}$ be parallel 2-form on M and write $\eta = \frac{f^*\overline{\omega}}{d\mu}$. Then η evolves by*

$$\begin{aligned}
\frac{\partial}{\partial t}\eta = {} & \lambda^2\triangle\eta + \lambda^2\left(\eta|A|^2 - 2\overline{\omega}(\nu_1, \nu_2)(h_{3k1}h_{4k2} - h_{4k1}h_{3k2})\right) \\
& + 2\lambda(\partial_1\lambda)\,\overline{\omega}(H, \partial_2) + 2\lambda(\partial_2\lambda)\,\overline{\omega}(\partial_1, H) \\
& + \lambda(\partial_k\lambda)\,\overline{\omega}(\overline{\nabla}_{\partial_1}\partial_k, \partial_2) + \lambda(\partial_k\lambda)\,\overline{\omega}(\partial_1, \overline{\nabla}_{\partial_2}\partial_k) \\
& - \lambda^2\left(\overline{\omega}((\overline{R}(\partial_k, \partial_1)\partial_k)^N, \partial_2) - \overline{\omega}((\overline{R}(\partial_k, \partial_2)\partial_k)^N, \partial_1)\right),
\end{aligned}$$

where we are summing k from 1 to 2.

Proof We will give only a sketch of the proof, since the argument is similar to that of Proposition 2.1 in [Wa1]. Calculate

$$\begin{aligned}
\frac{\partial}{\partial t}f^*\overline{\omega}(\partial_1, \partial_2) &= \overline{\omega}(\overline{\nabla}_{\lambda^2 H + \nabla(\frac{\lambda^2}{2})}\partial_1, \partial_2) + \overline{\omega}(\partial_1, \overline{\nabla}_{\lambda^2 H + \nabla(\frac{\lambda^2}{2})}\partial_2) \\
&= \lambda^2\overline{\omega}(\overline{\nabla}_{\partial_1}H, \partial_2) + \lambda^2\overline{\omega}(\partial_1, \overline{\nabla}_{\partial_2}H) \\
&\quad + 2\lambda(\partial_1\lambda)\,\overline{\omega}(H, \partial_2) + 2\lambda(\partial_2\lambda)\,\overline{\omega}(\partial_1, H) \\
&\quad + \overline{\omega}(\overline{\nabla}_{\partial_1}(\nabla(\frac{\lambda^2}{2})), \partial_2) + \overline{\omega}(\partial_1, \overline{\nabla}_{\partial_2}(\nabla(\frac{\lambda^2}{2}))) \\
&= \lambda^2\overline{\omega}(\overline{\nabla}_{\partial_1}H, \partial_2) + \lambda^2\overline{\omega}(\partial_1, \overline{\nabla}_{\partial_2}H) \\
&\quad + 2\lambda(\partial_1\lambda)\,\overline{\omega}(H, \partial_2) + 2\lambda(\partial_2\lambda)\,\overline{\omega}(\partial_1, H) + \eta\triangle(\frac{\lambda^2}{2}) \\
&\quad + \lambda(\partial_k\lambda)\overline{\omega}(\overline{\nabla}_{\partial_1}\partial_k, \partial_2) + \lambda(\partial_k\lambda)\overline{\omega}(\partial_1, \overline{\nabla}_{\partial_2}\partial_k). \qquad (3.2)
\end{aligned}$$

But from [Wa1], at p,

$$\begin{aligned}
\triangle\eta = {} & \overline{\omega}(\overline{\nabla}_{\partial_1}H, \partial_2) - \overline{\omega}(\overline{\nabla}_{\partial_2}H, \partial_1) + \eta|H|^2 \\
& + \overline{\omega}((\overline{R}(\partial_k, \partial_1)\partial_k)^N, \partial_2) - \overline{\omega}((\overline{R}(\partial_k, \partial_2)\partial_k)^N, \partial_1) \\
& - \eta|A|^2 + 2\overline{\omega}(\nu_1, \nu_2)(h_{3k1}h_{4k2} - h_{4k1}h_{3k2}). \qquad (3.3)
\end{aligned}$$

Then the lemma follows from (3.2), (3.3) and the evolution of the volume form. Q.E.D.

We introduce the following notation: for functions A and B, write $A \approx B$ if, at the point p,

$$|A - B| \leq C(Q + |\nabla\eta_3| + |\nabla(\eta_2 - 1/\lambda)|),$$

for some constant C depending on f_t. Then observe that $\eta_3 \approx 0 \approx a$ and $\xi_2^2 \approx 1 - \eta_2^2 \approx 1 - 1/\lambda^2$. Also, if $h_{ijk} = \overline{g}(K\partial_i, \overline{\nabla}_{\partial_j}\partial_k)$ then $h_{3jk} \approx h_{1jk}$, $h_{4jk} \approx h_{2jk}$ and

$$h_{ijk} \approx h_{jik} \approx h_{jki}.$$

We have the following lemma.

Lemma 3.2

(i) $\dfrac{\partial}{\partial t}\eta_3 \approx \lambda^2 \triangle \eta_3$

(ii) $\dfrac{\partial}{\partial t}(\eta_2 - 1/\lambda) \approx \lambda^2 \triangle (\eta_2 - 1/\lambda)$.

Proof First write $H = H_i dx^i$ where $H_i = g^{jk}h_{ijk}$. Then note that

$$\nabla\lambda \approx -\lambda^2 \xi_2 H, \tag{3.4}$$

To see this, calculate at p

$$\begin{aligned}
\partial_i\lambda &\approx -\lambda^2\overline{\omega}_2(\overline{\nabla}_{\partial_i}\partial_1, \partial_2) - \lambda^2\overline{\omega}_2(\partial_1, \overline{\nabla}_{\partial_i}\partial_2) \\
&\approx -\lambda^2\xi_2 h_{3i1} - \lambda^2\xi_2 h_{4i2} \\
&\approx -\lambda^2\xi_2 H_i.
\end{aligned}$$

For (i) we will make use of Lemma 3.1 with $\overline{\omega} = \overline{\omega}_3$. Observe that

$$\begin{aligned}
&2\lambda(\partial_1\lambda)\overline{\omega}_3(H, \partial_2) + 2\lambda(\partial_2\lambda)\overline{\omega}_3(\partial_1, H) \\
&\approx 2\lambda^3\xi_2 H_1\xi_3 H_2 - 2\lambda^3\xi_2 H_2\xi_3 H_1 = 0,
\end{aligned}$$

and

$$\begin{aligned}
&\lambda(\partial_k\lambda)\overline{\omega}_3(\overline{\nabla}_{\partial_1}\partial_k, \partial_2) + \lambda(\partial_k\lambda)\overline{\omega}_3(\partial_1, \overline{\nabla}_{\partial_2}\partial_k) \\
&\approx -\lambda(\partial_k\lambda)h_{41k}\xi_3 + \lambda(\partial_k\lambda)h_{32k}\xi_3 \approx 0.
\end{aligned}$$

We need to deal now with the curvature terms. First observe that the Ricci curvature can be approximated by

$$\overline{\mathrm{Ric}}(KX, Y) \approx \frac{1}{2}\overline{R}(X, Y, e_A, Ke_A),$$

where we are summing over the orthonormal basis e_A for TM. Then

$$\begin{aligned}
&\overline{\omega}_3((\overline{R}(\partial_k, \partial_1)\partial_k)^N, \partial_2) - \overline{\omega}_3((\overline{R}(\partial_k, \partial_2)\partial_k)^N, \partial_1) \\
&= -\xi_3\overline{R}(\nu_2, \partial_k, \partial_k, \partial_1) + \xi_3\overline{R}(\nu_1, \partial_k, \partial_k, \partial_2) \\
&\approx -\overline{\mathrm{Ric}}(K\partial_1, \partial_2) = 0,
\end{aligned}$$

and (i) follows. For (ii), calculate:

$$|A|^2 + 2(h_{3k1}h_{4k2} - h_{4k1}h_{3k2}) \approx |H|^2,$$

and

$$2\lambda(\partial_1\lambda)\overline{\omega}_2(H,\partial_2)+2\lambda(\partial_2\lambda)\overline{\omega}_2(\partial_1,H)\approx -2\lambda(\lambda^2-1)|H|^2,$$

and

$$\lambda(\partial_k\lambda)\overline{\omega}_2(\overline{\nabla}_{\partial_1}\partial_k,\partial_2)+\lambda(\partial_k\lambda)\overline{\omega}_2(\partial_1,\overline{\nabla}_{\partial_2}\partial_k)\approx -\lambda(\lambda^2-1)|H|^2.$$

Then since the curvature terms can be neglected as in (i), we see that

$$\frac{\partial}{\partial t}\eta_2\approx \lambda^2\triangle\eta_2+\lambda|H|^2-3\lambda(\lambda^2-1)|H|^2.$$

But a short calculation shows that

$$\frac{\partial}{\partial t}\left(\frac{1}{\lambda}\right)=\lambda^2\triangle\left(\frac{1}{\lambda}\right)+\lambda|H|^2-3\frac{|\nabla\lambda|^2}{\lambda},$$

and (ii) follows after making use of (3.4). Q.E.D.

The inequality (3.1) and hence Theorem 1 now follow easily. Indeed, writing $\frac{\partial}{\partial t}\eta_3=\lambda^2\triangle\eta_3+E_1$ and $\frac{\partial}{\partial t}(\eta_2-1/\lambda)=\lambda^2\triangle(\eta_2-1/\lambda)+E_2$ for $E_1\approx 0\approx E_2$ we see that

$$\begin{aligned}\frac{\partial Q}{\partial t}&=\lambda^2\triangle Q-2\lambda^2|\nabla(\eta_2-\frac{1}{\lambda})|^2-2\lambda^2|\nabla\eta_3|^2+2E_1(\eta_2-\frac{1}{\lambda})+2E_2\eta_3\\&\leq \lambda^2\triangle Q+CQ,\end{aligned}$$

where we have made use of the inequality $2ab\leq(a^2+b^2)$ for $a,b\in\mathbb{R}$. Q.E.D.

4. Evolution equations for the Special H-flow

In this section we will give the proof of Theorem 2. As before, let $\{x^1,x^2\}$ be a normal coordinate system at a point p for S with $\rho(\partial_1,\partial_2)>0$. Define normal vectors $\nu_i=K\partial_i$ for $i=1,2$. Then $\{\partial_1,\partial_{2,},\nu_1,\nu_2\}$ gives an basis for the tangent bundle of M and is orthonormal at p. Note that at the point p, we have $(\overline{\nabla}_{\partial_i}\nu_j)^N=0$. In this coordinate system, the second fundamental form is given by $h_{ijk}=\overline{g}(\nu_i,\overline{\nabla}_{\partial_j}\partial_k)$, and we define the mean curvature 1-form H_idx^i by $H_i=g^{jk}h_{ijk}$. The second fundamental form is fully symmetric:

$$h_{ijk}=h_{jik}=h_{jki}.$$

Recall from section 3 that η_1 is defined by

$$\eta_1=\frac{f^*(\overline{\omega}_1)}{d\mu}=\frac{N_1}{\lambda}$$

We can write down expressions now for the almost complex structures in terms of η_1.

Lemma 4.1 *At p, I, J and K are given by*

$$I=\begin{pmatrix}0&\eta_1&0&\sqrt{1-\eta_1^2}\\-\eta_1&0&-\sqrt{1-\eta_1^2}&0\\0&\sqrt{1-\eta_1^2}&0&-\eta_1\\-\sqrt{1-\eta_1^2}&0&\eta_1&0\end{pmatrix},$$

$$J = \begin{pmatrix} 0 & \sqrt{1-\eta_1^2} & 0 & -\eta_1 \\ -\sqrt{1-\eta_1^2} & 0 & \eta_1 & 0 \\ 0 & -\eta_1 & 0 & -\sqrt{1-\eta_1^2} \\ \eta_1 & 0 & \sqrt{1-\eta_1^2} & 0 \end{pmatrix}$$

and

$$K = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix},$$

Proof It is trivial to see that K takes the above form. By definition, ${I_1}^2 = \eta_1$. Since

$$\rho(\partial_1, \partial_2) = (f^* \overline{\omega}_2)(\partial_1, \partial_2) = \overline{g}(J\partial_1, \partial_2),$$

and $\rho(\partial_1, \partial_2) > 0$, we have that ${J_1}^2 > 0$. Hence ${J_1}^2 = \sqrt{1-\eta_1^2}$, and we may write

$$J = \begin{pmatrix} 0 & \sqrt{1-\eta_1^2} & 0 & a\eta_1 \\ -\sqrt{1-\eta_1^2} & 0 & b\eta_1 & 0 \\ 0 & -b\eta_1 & 0 & c\sqrt{1-\eta_1^2} \\ -a\eta_1 & 0 & -c\sqrt{1-\eta_1^2} & 0 \end{pmatrix},$$

where a, b and c are each either 1 or -1. Now calculate

$$\begin{aligned} I &= JK \\ &= \begin{pmatrix} 0 & \sqrt{1-\eta_1^2} & 0 & a\eta_1 \\ -\sqrt{1-\eta_1^2} & 0 & b\eta_1 & 0 \\ 0 & -b\eta_1 & 0 & c\sqrt{1-\eta_1^2} \\ -a\eta_1 & 0 & -c\sqrt{1-\eta_1^2} & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix} \\ &= \begin{pmatrix} 0 & -a\eta_1 & 0 & \sqrt{1-\eta_1^2} \\ -b\eta_1 & 0 & -\sqrt{1-\eta_1^2} & 0 \\ 0 & -c\sqrt{1-\eta_1^2} & 0 & -b\eta_1 \\ c\sqrt{1-\eta_1^2} & 0 & -a\eta_1 & 0 \end{pmatrix}, \end{aligned}$$

from which we obtain $a = -1$, $b = 1$ and $c = -1$. This completes the proof.

A short calculation shows that, at the point p,

$$\partial_k \lambda = \lambda^2 \eta_1 H_k.$$

Then part (i) of Theorem 2 follows from the evolution of λ and the maximum principle.

We now calculate the evolution of the second fundamental form. We will use the convention that an underlined index denotes the application of the complex structure K, so that, for example:

$$\overline{R}_{lj\underline{i}k} = \overline{g}(\overline{R}(\partial_l, \partial_j)\partial_k, K\partial_i).$$

Lemma 4.2 *We have the following formulae for the Special H-flow:*

(i) The second fundamental form evolves by

$$\begin{aligned}\frac{\partial}{\partial t}h_{ijk} = \lambda^2 \{ & \triangle h_{ijk} + (3\lambda^2-2)\left(H_iH_lh_{jkl} + H_jH_lh_{kil} + H_kH_lh_{ijl}\right.\\ & \left.+2H_iH_jH_k\right) + \lambda\eta_1\left(H_{l,i}h_{jkl} + H_{l,j}h_{kil} + H_{l,k}h_{ijl}\right.\\ & \left.+ 2H_iH_{j,k} + 2H_jH_{k,i} + 2H_kH_{i,j} + H_lh_{ijk,l}\right)\\ & - \left(H_mh_{mli}h_{ljk} + H_mh_{mlj}h_{lki} + H_mh_{mlk}h_{lij}\right)\\ & + 2\overline{R}_{limj}h_{klm} + 2\overline{R}_{ljmk}h_{ilm} + 2\overline{R}_{lkmi}h_{jlm}\\ & - \overline{R}_{lilm}h_{jkm} - \overline{R}_{ljlm}h_{kim} - \overline{R}_{lklm}h_{ijm}\\ & + h_{lmr}(h_{ijl}h_{kmr} + h_{jkl}h_{imr} + h_{kil}h_{jmr}) - 2h_{ilm}h_{jmr}h_{krl}\\ & -(\overline{\nabla}_{\partial_j}\overline{R})_{kll\underline{i}} - (\overline{\nabla}_{\partial_l}\overline{R})_{jlk\underline{i}}\} . \end{aligned} \tag{4.1}$$

(ii) The square of the norm of the second fundamental form evolves by

$$\begin{aligned}\frac{\partial}{\partial t}|A|^2 = \lambda^2 \{ & \triangle|A|^2 - 2|\nabla A|^2 + 4(3\lambda^2-2)H_iH_jH_kh_{ijk} + 12\lambda\eta_1H_iH_{j,k}h_{ijk}\\ & + 2\lambda\eta_1H_lh_{ijk,l}h_{ijk} + 12\overline{R}_{lkmi}h_{jlm}h_{ijk} - 6\overline{R}_{lilm}h_{jkm}h_{ijk}\\ & + 6h_{lmr}h_{ijl}h_{kmr}h_{ijk} - 4h_{ilm}h_{jmr}h_{krl}h_{ijk} - 2(\overline{\nabla}_{\partial_j}\overline{R})_{kll\underline{i}}h_{ijk}\\ & -2(\overline{\nabla}_{\partial_l}\overline{R})_{jlk\underline{i}}h_{ijk}\} . \end{aligned} \tag{4.2}$$

(iii) The square of the norm of the mean curvature evolves by

$$\begin{aligned}\frac{\partial}{\partial t}|H|^2 = \lambda^2 \{ & \triangle|H|^2 - 2|\nabla H|^2 + 4(3\lambda^2-2)|H|^4 + 10\lambda\eta_1H_iH_jH_{i,j}\\ & + 4\lambda\eta_1|H|^2H_{i,i} + 2h_{ilm}h_{jlm}H_iH_j - 2\overline{R}_{lklm}H_mH_k\\ & -2H_k(\overline{\nabla}_{\partial_i}\overline{R})_{kll\underline{i}} - 2H_k(\overline{\nabla}_{\partial_l}\overline{R})_{ilk\underline{i}}\} . \end{aligned} \tag{4.3}$$

Proof Calculate at p,

$$\begin{aligned}\frac{\partial}{\partial t}h_{ijk} &= (\nabla(\frac{\lambda^2}{2}) + \lambda^2H)(\overline{g}(\overline{\nabla}_{\partial_j}\partial_k, \nu_i))\\ &= \overline{g}(\overline{\nabla}_{\partial_j}\overline{\nabla}_{\nabla(\frac{\lambda^2}{2})+\lambda^2H}\partial_k, \nu_i) + \overline{g}(\overline{R}(\nabla(\frac{\lambda^2}{2}) + \lambda^2H, \partial_j)\partial_k, \nu_i)\\ &\quad + \overline{g}(\overline{\nabla}_{\partial_j}\partial_k, \overline{\nabla}_{\nabla(\frac{\lambda^2}{2})+\lambda^2H}\nu_i)\\ &= \overline{g}(\overline{\nabla}_{\partial_j}\overline{\nabla}_{\partial_k}(\nabla(\frac{\lambda^2}{2}) + \lambda^2H), \nu_i) + \overline{g}(\overline{R}(\nabla(\frac{\lambda^2}{2}), \partial_j)\partial_k, \nu_i)\\ &\quad + \lambda^2\overline{g}(\overline{R}(H, \partial_j)\partial_k, \nu_i) - \overline{g}(K\overline{\nabla}_{\partial_j}\partial_k, \overline{\nabla}_{\partial_i}(\nabla(\frac{\lambda^2}{2}) + \lambda^2H)). \end{aligned} \tag{4.4}$$

Now calculate

$$\begin{aligned}\overline{g}(\overline{\nabla}_{\partial_j}\overline{\nabla}_{\partial_k}(\nabla(\lambda^2)), \nu_i) &= \overline{g}(\partial_j\partial_l(\lambda^2)\overline{\nabla}_{\partial_k}\partial_l, \nu_i) + \overline{g}(\partial_k\partial_l(\lambda^2)\overline{\nabla}_{\partial_j}\partial_l, \nu_i)\\ &\quad + \overline{g}(\partial_l(\lambda^2)\overline{\nabla}_{\partial_j}\overline{\nabla}_{\partial_k}\partial_l, \nu_i)\\ &= \partial_j\partial_l(\lambda^2)h_{ikl} + \partial_k\partial_l(\lambda^2)h_{ijl} + \partial_l(\lambda^2)h_{ikl,j}, \end{aligned} \tag{4.5}$$

and

$$\overline{g}(\overline{\nabla}_{\partial_j}\overline{\nabla}_{\partial_k}(\lambda^2 H),\nu_i) = \partial_j\partial_k(\lambda^2)H_i + \partial_k(\lambda^2)H_{i,j} + \partial_j(\lambda^2)H_{i,k} + \lambda^2\overline{g}(\overline{\nabla}_{\partial_j}\overline{\nabla}_{\partial_k}H,\nu_i). \tag{4.6}$$

For the last term, we follow the calculation in [Wa1], p. 332. First,

$$\overline{g}(\overline{\nabla}_{\partial_j}\overline{\nabla}_{\partial_k}H,\nu_i) = \overline{g}(\overline{\nabla}_{\partial_j}(\overline{\nabla}_{\partial_k}H)^T,\nu_i) + \overline{g}(\overline{\nabla}_{\partial_j}(\overline{\nabla}_{\partial_k}H)^N,\nu_i).$$

We want to calculate the first term on the right hand side of this equation. To begin with, write

$$\overline{g}((\overline{\nabla}_{\partial_k}H)^T,\partial_l) = \overline{g}(\overline{\nabla}_{\partial_k}H,\partial_l) = -\overline{g}(H,\overline{\nabla}_{\partial_k}\partial_l) = -H_m h_{mkl},$$

and so

$$\overline{g}(\overline{\nabla}_{\partial_j}(\overline{\nabla}_{\partial_k}H)^T,\nu_i) = -\overline{g}((\overline{\nabla}_{\partial_k}H)^T,\overline{\nabla}_{\partial_j}\nu_i) = -H_m h_{mkl}h_{ijl}.$$

Hence

$$\overline{g}(\overline{\nabla}_{\partial_j}\overline{\nabla}_{\partial_k}H,\nu_i) = H_{i,kj} - H_m h_{mkl}h_{ijl}. \tag{4.7}$$

The last term of (4.4) is given by

$$-\overline{g}(K\overline{\nabla}_{\partial_j}\partial_k,\overline{\nabla}_{\partial_i}(\nabla(\frac{\lambda^2}{2}) + \lambda^2 H)) = \frac{1}{2}\partial_i\partial_l(\lambda^2)h_{jkl} - \lambda^2 H_l h_{jkm}h_{ilm}. \tag{4.8}$$

Inserting (4.5), (4.6), (4.7) and (4.8) into (4.4), and making use of the Codazzi equation, we have

$$\begin{aligned}\frac{\partial}{\partial t}h_{ijk} = {} & \lambda^2 H_{i,kj} + \frac{1}{2}(\partial_j\partial_l(\lambda^2)h_{ikl} + \partial_k\partial_l(\lambda^2)h_{ijl} + \partial_i\partial_l(\lambda^2)h_{jkl}) \\ & + \partial_j\partial_k(\lambda^2)H_i + \partial_k(\lambda^2)H_{i,j} + \partial_j(\lambda^2)H_{i,k} + \frac{1}{2}\partial_l(\lambda^2)h_{ijk,l} \\ & - \lambda^2(H_m h_{mlk}h_{ijl} + H_m h_{mli}h_{jkl}) + \lambda^2 H_l\overline{R}_{\underline{l}j\underline{i}k}.\end{aligned} \tag{4.9}$$

Recall that at the point p,

$$\partial_k(\lambda^2) = 2\lambda^3\eta_1 H_k. \tag{4.10}$$

A straightforward calculation shows that

$$\partial_j\partial_l(\lambda^2) = 2\lambda^2(3\lambda^2-2)H_jH_l + 2\lambda^3\eta_1 H_{l,j}, \tag{4.11}$$

where we are using the fact that $\eta_1^2 = 1 - 1/\lambda^2$. In (4.9), we obtain

$$\begin{aligned}\frac{\partial}{\partial t}h_{ijk} = {} & \lambda^2\{H_{i,kj} + (3\lambda^2-2)(H_iH_lh_{jkl} + H_jH_lh_{ikl} + H_kH_lh_{ijl} + 2H_iH_jH_k) \\ & + \lambda\eta_1(H_{l,i}h_{jkl} + H_{l,j}h_{kil} + H_{l,k}h_{ijl} + 2H_iH_{j,k} + 2H_jH_{k,i} + 2H_kH_{i,j} \\ & + H_lh_{ijk,l}) - (H_mh_{mlk}h_{ijl} + H_mh_{mli}h_{jkl}) + H_l\overline{R}_{\underline{l}j\underline{i}k}\}.\end{aligned} \tag{4.12}$$

We now use the formula for the Laplacian of the second fundamental form:

$$\begin{aligned}\triangle h_{ijk} = {} & H_{i,kj} + (\overline{\nabla}_{\partial_j}\overline{R})_{kll\underline{i}} + (\overline{\nabla}_{\partial_l}\overline{R})_{jlk\underline{i}} \\ & - 2\overline{R}_{lkmi}h_{jlm} - 2\overline{R}_{limj}h_{klm} - 2\overline{R}_{ljmk}h_{ilm} + \overline{R}_{l\underline{j}\underline{i}k}H_l \\ & + \overline{R}_{lilm}h_{jkm} + \overline{R}_{ljlm}h_{kim} + \overline{R}_{lklm}h_{ijm} - h_{jkl}\overline{R}_{il} \\ & + h_{ikm}(h_{lmj}H_l - h_{lmr}h_{ljr}) + h_{iml}(h_{rmj}h_{rkl} - h_{rml}h_{rkj}) \\ & + h_{mkr}(h_{mlj}h_{ilr} - h_{mlr}h_{ilj}).\end{aligned}$$

Substituting into (4.12) gives (4.1).

We now calculate the evolution of the square of the second fundamental form,

$$|A|^2 = g^{ip}g^{jq}g^{kr}h_{ijk}h_{pqr}.$$

First, at a point p we have, using (4.11),

$$\begin{aligned}\frac{\partial}{\partial t}g^{ip} &= -\nabla_i\nabla_p(\lambda^2) + 2\lambda^2 H_m h_{mip} \\ &= -2\lambda^2(3\lambda^2-2)H_iH_p - 2\lambda^3\eta_1 H_{i,p} + 2\lambda^2 H_m h_{mip}.\end{aligned} \tag{4.13}$$

Then

$$\frac{\partial}{\partial t}|A|^2 = 2\left(\frac{\partial}{\partial t}h_{ijk}\right)h_{ijk} - 6\lambda^2\left((3\lambda^2-2)H_iH_p + \lambda\eta_1 H_{i,p} - H_m h_{mip}\right)h_{ijk}h_{pjk}.$$

Making use of (4.1), we obtain (4.2) after some simplification.

Finally, we calculate the evolution of the square of the mean curvature. Using (4.13),

$$\begin{aligned}\frac{\partial}{\partial t}|H|^2 &= \frac{\partial}{\partial t}(g^{kl}g^{ij}h_{ijk}g^{pq}h_{pql}) \\ &= \left(\frac{\partial}{\partial t}g^{kl}\right)H_kH_l + 2\left(\frac{\partial}{\partial t}g^{ij}\right)h_{ijk}H_k + 2g^{ij}H_k\frac{\partial}{\partial t}h_{ijk} \\ &= -2\lambda^2(3\lambda^2-2)|H|^4 - 2\lambda^3\eta_1 H_{l,k}H_kH_l + 2\lambda^2 H_m h_{mkl}H_kH_l \\ &\quad - 4\lambda^2(3\lambda^2-2)H_iH_jH_kh_{ijk} - 4\lambda^3\eta_1 H_{j,i}h_{ijk}H_k \\ &\quad + 4\lambda^2 H_m h_{mij}h_{ijk}H_k + 2g^{ij}H_k\frac{\partial}{\partial t}h_{ijk}.\end{aligned}$$

Then from (4.1), we obtain (4.3), after some canceling. This finishes the proof of the lemma. Q.E.D.

Proof of Theorem 2 From the evolution equations of $|A|^2$, λ and g, by a similar argument to [Hu1], [Hu2], we obtain for constants $C(k)$,

$$\frac{\partial}{\partial t}|\nabla^k A|^2 \le \lambda^2\triangle|\nabla^k A|^2 - 2\lambda^2|\nabla^k A|^2 + C(k)\sum_{a+b+c=k}|\nabla^a A|\,|\nabla^b A|\,|\nabla^c A|\,|\nabla^k A|,$$

and (ii) follows by an induction argument. Part (iii) can be proved using a straightforward modification of the argument of [Hu2], section 8. Q.E.D.

5. Singularity formation

In this section we give the proof of Theorem 3. Suppose that there is a Type I* singularity at time T. We will show that this gives a contradiction using a monotonicity formula. Fix $y_0 \in M$. Let $\iota : M \to \mathbb{R}^N$ be an isometric embedding and suppose that y_0 corresponds to the origin in $\mathbb{R}^N$. Write $H \in TM/TS$ for the mean curvature of S in M and $\tilde{H} \in T\mathbb{R}^N/TS$ for the mean curvature of S in $\mathbb{R}^N$. Set $E = H - \tilde{H}$.

Let $\Phi : \mathbb{R}^N \times [0,T) \to \mathbb{R}$ be the 'backwards heat kernel' on $\mathbb{R}^N$ given by

$$\Phi(y,t) = \frac{1}{4\pi\lambda_0^2(T-t)} \exp\left(-\frac{|y|^2}{4\lambda_0^2(T-t)}\right).$$

Observe that

$$\overline{\nabla}^{\mathbb{R}^N}\Phi = -\frac{\Phi}{2\lambda_0^2(T-t)}y.$$

Restricted to S, Φ evolves by

$$\begin{aligned}&\frac{\partial}{\partial t}\Phi + \lambda_0^2 \triangle\Phi \\ &= -\Phi\left[\frac{|f^\perp|^2}{4\lambda_0^2(T-t)^2} + \frac{\lambda^2 H\cdot f^\perp}{2\lambda_0^2(T-t)} + \frac{\lambda\nabla\lambda\cdot f^T}{2\lambda_0^2(T-t)} + \frac{f^\perp\cdot\tilde{H}}{2(T-t)}\right],\end{aligned}$$

where f^T and $f^\perp$ are the tangential and normal components to S of the function f, considered as a vector in $\mathbb{R}^N$. For $s \geq 1$, calculate

$$\begin{aligned}&\frac{d}{dt}\int_S \lambda^s\Phi d\mu \\ &= \int_S \Big\{ s(\lambda_0^2 - \lambda^2)\lambda^{s-1}\nabla\lambda\cdot\nabla\Phi - (s+1)s\lambda^s|\nabla\lambda|^2\Phi \\ &\quad - (s+1)\lambda^{s+2}\Phi|H|^2 - \frac{1}{2}\lambda^s\Phi\left|\frac{f^\perp}{2\lambda_0(T-t)} + \lambda_0(1+\frac{\lambda^2}{\lambda_0^2})H\right|^2 \\ &\quad + \frac{\lambda^s\Phi f^\perp\cdot E}{2(T-t)} - \frac{\lambda^s\Phi|f^\perp|^2}{8\lambda_0^2(T-t)^2} + \frac{1}{2}\Phi\lambda^s\lambda_0^2(1+\frac{\lambda^2}{\lambda_0^2})^2|H|^2\Big\} d\mu. \qquad (5.1)\end{aligned}$$

Now pick s large enough so that

$$(s+1)\lambda^{s+2} > 1 + \frac{1}{2}\lambda^s\lambda_0^2(1+\frac{\lambda^2}{\lambda_0^2})^2 \qquad (5.2)$$

To deal with the first term of (5.1), we make use of (1.4) and the definition of Φ to see that

$$\begin{aligned}&\int_S s|\lambda_0^2 - \lambda^2|\lambda^{s-1}|\nabla\lambda\cdot\nabla\Phi| d\mu \\ &\leq \int_S s\lambda^s|\nabla\lambda|^2\Phi d\mu + C\int_{|f|^2\leq(T-t)^{1-\delta/2}} (\lambda-\lambda_0)\frac{|f^T|^2}{(T-t)^2}\Phi d\mu \\ &\quad + C\int_{|f|^2>(T-t)^{1-\delta/2}} \frac{|f^T|^2}{(T-t)^2}\Phi d\mu \\ &\leq \int_S s\lambda^s|\nabla\lambda|^2\Phi d\mu + \frac{C}{(T-t)^{1-\delta/4}}. \qquad (5.3)\end{aligned}$$

Finally, observe that

$$\frac{\lambda^s \Phi f^\perp \cdot E}{2(T-t)} \leq \frac{\lambda^s \Phi |f^\perp|^2}{8\lambda_0^2 (T-t)^2} + C\Phi, \tag{5.4}$$

since E is bounded. Then combining (5.1),(5.2), (5.3) and (5.4) we obtain

$$\begin{aligned} &\frac{d}{dt}\int_S \lambda^s \Phi d\mu \\ &\leq \frac{C}{(T-t)^{1-\delta/4}} - \int_S \Phi \left\{ |H|^2 + \frac{1}{2}\lambda^s \left| \frac{f^\perp}{2\lambda_0 (T-t)} + \lambda_0 (1 + \frac{\lambda^2}{\lambda_0^2}) H \right|^2 \right\} d\mu. \end{aligned}$$

It follows that $\lim_{t\to T} \int_S \lambda^s \Phi d\mu$ exists. We can now use a blow-up argument. First, observe that if the flow has a singularity at time T then there exists a constant $C > 0$ such that

$$\sup_S |A|^2 \geq \frac{1}{C(T-t)}, \tag{5.5}$$

for all t sufficiently close to T. Indeed if we set $w = \sup_S |A|^2$ then by Lemma 4.2 we have

$$\frac{\partial w}{\partial t} \leq C'(w^2 + 1),$$

and hence

$$\tan^{-1}(w(t')) - \tan^{-1}(w(t)) \leq C'(t'-t), \text{ for } t < t' < T.$$

Letting $t' \to T$ we see that

$$w(t) \geq \tan(\pi/2 - C'(T-t)) \geq \frac{1}{C(T-t)},$$

giving (5.5).

Pick a sequence of points and times $(x_k, t_k) \in S \times [0, T)$ with $t_k \to T$ and $x_k \to x \in S$ so that

$$\nu_k^2 = \sup_{x\in S,\ t\in[0,t_k]} |A|(x,t) = |A|^2(x_k, t_k).$$

We then perform a parabolic scaling around the point $y_0 = f(x) \in M \subset \mathbb{R}^N$ by dilating space by a factor ν_k^2 and time by a factor ν_k. The second fundamental form is bounded and by the estimates of Theorem 2 (cf. [Hu2]) so are its covariant derivatives. The rescaled flows converge smoothly and have nonzero second fundamental form by (5.5). But from the monotonicity formula above we see (cf. the argument in [Wa1]) that the blow-up limit satisfies $H = 0$ and $f^\perp = 0$ and hence is a plane. This is a contradiction. Q.E.D.

6. Convergence of the flow

Let $\overline{\zeta}$ be a parallel anti-self-dual calibrating form on M so that $\overline{\omega}_2^2$ and $\overline{\zeta}^2$ determine opposite orientations on M. Write $\beta_1 = 1/\lambda = f^*(\overline{\omega}_2)/d\mu$ and $\beta_2 = f^*\overline{\zeta}/d\mu$. We can pick normal coordinates $\{x^1, x^2\}$ on S and local sections of the normal bundle e_3, e_4 such that, writing $e_i = f_*\partial_i$ for $i = 1, 2$, we have at a point p:

$$(\overline{\omega}_2(e_A, e_B)) = \begin{pmatrix} 0 & \beta_1 & \gamma_1 & 0 \\ -\beta_1 & 0 & 0 & -\gamma_1 \\ -\gamma_1 & 0 & 0 & \beta_1 \\ 0 & \gamma_1 & -\beta_1 & 0 \end{pmatrix}$$

with $\beta_1^2 + \gamma_1^2 = 1$, and

$$(\overline{\zeta}(e_A, e_B)) = \begin{pmatrix} 0 & \beta_2 & \gamma_2 & 0 \\ -\beta_2 & 0 & 0 & \gamma_2 \\ -\gamma_2 & 0 & 0 & -\beta_2 \\ 0 & -\gamma_2 & \beta_2 & 0 \end{pmatrix},$$

with $\beta_2^2 + \gamma_2^2 = 1$ (see [Wa1], Section 3). By a straightforward calculation, we have the following lemma.

Lemma 6.1 *In the above coordinates, β_1 and β_2 evolve by:*

$$\begin{aligned} \frac{\partial}{\partial t}\beta_1 &= \lambda^2 \{ \triangle\beta_1 + \beta_1(|A|^2 - 2h_{3k1}h_{4k2} + 2h_{41k}h_{3k2}) - \lambda\gamma_1^2(2(h_{411} + h_{312})H_4 \\ &\qquad + 2(h_{421} + h_{322})H_3 + \sum_{k=1}^{2}(h_{41k} + h_{32k})^2) \}, \end{aligned} \tag{6.1}$$

and

$$\begin{aligned} \frac{\partial}{\partial t}\beta_2 &= \lambda^2 \{ \triangle\beta_2 + \beta_2(|A|^2 + 2h_{3k1}h_{4k2} - 2h_{41k}h_{3k2}) + \lambda\gamma_1\gamma_2(2(h_{411} + h_{312})H_4 \\ &\qquad - 2(h_{421} + h_{322})H_3 + \sum_{k=1}^{2} h_{41k}^2 - \sum_{k=1}^{2} h_{32k}^2) \}, \end{aligned} \tag{6.2}$$

where $h_{\alpha ij} = \overline{g}(e_\alpha, \overline{\nabla}_{\partial_i}\partial_j)$.

Set $\mu = \beta_1 + \beta_2$. Initially, $\beta_1 > 1 - \epsilon$, $\beta_2 > 1 - \epsilon$ and hence γ_1, γ_2 are of order ϵ. It follows from Lemma 6.1 and a maximum principle argument that if ϵ is sufficiently small then the minimum of μ is nondecreasing along the flow. It follows that

$$\frac{\partial}{\partial t}\mu \geq \lambda^2(\triangle\mu + c_1(\epsilon)\mu|A|^2), \tag{6.3}$$

for some constant $c_1(\epsilon)$ which tends to 1 as ϵ tends to zero. We also have

$$|\nabla\mu|^2 \leq c_2(\epsilon)\mu^2|A|^2,$$

for some constant $c_2(\epsilon)$ which tends to zero as ϵ tends to zero. From Lemma 4.2,

$$\frac{\partial}{\partial t}|A|^2 \leq \lambda^2 \triangle |A|^2 + K_1|A|^4 + K_2, \tag{6.4}$$

for some constants K_1 and K_2. Then the argument in [Wa1], section 7, implies that $|A|^2$ is bounded along the flow for ϵ sufficiently small. By Theorem 2, the covariant derivatives of the second fundamental form are bounded and the flow exists for all time. From (6.3) and (6.4) we obtain for some constants C_1 and C_2,

$$\int_0^\infty \left(\int_S |A|^2 d\mu \right) dt \leq C_1 \quad \text{and} \quad \frac{d}{dt}\left(\int_S |A|^2 d\mu \right) \leq C_2.$$

Hence

$$\int_S |A|^2 d\mu \to 0.$$

Moreover, the maps f_t converge in L^1 and, by an argument similar to that in [La], the submanifolds $f(S)$ converge in C^∞ as local graphs to a smooth submanifold with zero second fundamental form. It follows that the maps f_t converge smoothly to a map f_∞ whose image is a totally geodesic complex curve in M. Q.E.D.

Acknowledgements The authors are very grateful to: Mu-Tao Wang for his excellent course at Columbia University on minimal submanifolds and for a number of useful discussions; our former advisor D.H. Phong for his continued help and advice; S.-T. Yau for his support and encouragement; and Simon Donaldson, Mark Haskins, Richard Thomas, Tom Ilmanen and Bill Minicozzi for some helpful discussions.

References

[ChLi1] Chen, J. and Li, J. *Mean curvature flow of surface in 4-manifolds*, Adv. Math. **163** (2001), no. 2, 287–309

[ChLi2] Chen, J. and Li, J. *Singularity of mean curvature flow of Lagrangian submanifolds*, Invent. Math. **156** (2004), no. 1, 25–51

[ChLiTi] Chen, J., Li, J. and Tian, G. *Two-dimensional graphs moving by mean curvature flow*, Acta Math. Sin. **18** (2002), no. 2, 209–224

[Ch] Chen, X.X. *A new parabolic flow in Kähler manifolds*, Comm. Anal. Geom. **12** (2004), no. 4, 837–852

[Do] Donaldson, S. K. *Moment maps and diffeomorphisms*, Asian J. Math. **3** (1999), no. 1, 1–16

[Hu1] Huisken, G. *Contracting convex hypersurfaces in Riemannian manifolds by their mean curvature*, Invent. Math. **84** (1986), 463–480

[Hu2] Huisken, G. *Asymptotic behavior for singularities of the mean curvature flow*, J. Differential Geom. **31** (1990), no. 1, 285-299

[La] Langer, J. *A compactness theorem for surfaces with L_p-bounded second fundamental form*, Math. Ann. **270** (1985), 223–234

[MiWo] Micallef, M. J. and Wolfson, J. *The second variation of minimal surfaces in 4-manifolds*, Math. Ann. **295** (1993), 245–267

[Ne] Neves, A. *Singularities of Lagrangian mean curvature flow*, preprint

[Sm1] Smoczyk, K. *A canonical way to deform a Lagrangian submanifold*, preprint, dg-ga/9605005

[Sm2] Smoczyk, K. *Angle theorems for the Lagrangian mean curvature flow*, Math. Z. **240** (2002), 849–883

[Sm3] Smoczyk, K. *Longtime existence of the Lagrangian mean curvature flow*, Calc. Var. Partial Differential Equations **20** (2004), no. 1, 25–46

[SmWa] Smoczyk, K. and Wang, M.-T. *Mean curvature flows of Lagrangian submanifolds with convex potentials*, J. Differential Geom. **62** (2002), no. 2, 243–257

[SoWe] Song, J. and Weinkove, B. *On the convergence and singularities of the J-flow with applications to the Mabuchi energy*, preprint, math.DG/0410418

[StYaZa] Strominger, A., Yau, S.-T. and Zaslow, F. *Mirror symmetry is T-duality*, Nucl. Phys. B **479** (1996), 243–259

[ThYa] Thomas, R. P. and Yau, S.-T. *Special Lagrangians, stable bundles and mean curvature flow*, Comm. Anal. Geom. **10** (2002), no.5, 1075–1113

[TsWa] Tsui, M.-P. and Wang, M.-T. *Mean curvature flows and isotopy of maps between spheres*, Comm. Pure Appl. Math. **57** (2004), no. 8, 1110–1126

[Wa1] Wang, M.-T. *Mean curvature flow of surfaces in Einstein four-manifolds*, J. Diff. Geom. **57** (2001), 301–338

[Wa2] Wang, M.-T. *Deforming area preserving diffeomorphism of surfaces by mean curvature flow*, Math. Res. Lett. **8** (2001), no. 5-6, 651–661

[Wa3] Wang, M.-T. *A convergence result of the Lagrangian mean curvature flow*, preprint, math.DG/0508354

[We] Weinkove, B. *On the J-flow in higher dimensions and the lower boundedness of the Mabuchi energy*, J. Diff. Geom. **73** (2006), no. 2, 351–358

[Zh] Zhu, X.-P. *Lectures on Mean Curvature Flows*, AMS/IP Studies in Advanced Mathematics, ed. S.-T. Yau, 2002